**Non-linear system of simultaneous equations**

**1.** Solve for real numbers $x,y$ :

 $\left\{\begin{array}{c}4x^{2}+y^{2}=17+4x\\\left(2x-1\right)^{2}+\left(y-8\right)^{2}=34\end{array}\right.$

 $\left\{\begin{array}{c}4x^{2}+y^{2}=17+4x ….(1)\\\left(2x-1\right)^{2}+\left(y-8\right)^{2}=34 ….(2)\end{array}\right.$

 From $(2)$, $4 x^{2}-4 x+1+y^{2}-16 y+64=34$

 $4 x^{2}+y^{2}-16 y+31=4 x ….(3)$

 $\left(1\right)-\left(3\right),$ $16 y-31=17$

 $∴ y=3 ….(4)$

 $\left(4\right)\downright \left(1\right),$ $4x^{2}+9=17+4x$

 $4x^{2}-4x-8=0$

$$∴ x=-1 or x=2$$

 The solution is $\left(x,y\right)=\left(-1,3\right) or \left(2,3\right)$

**2.** Solve for real numbers:

$$\left\{\begin{matrix}w+x+y+z=10\\w^{2}+x^{2}+y^{2}+z^{2}=30\\w^{3}+x^{3}+y^{3}+z^{2}=100\\wxyz=24\end{matrix}\right.$$

 By inspection, $\left(w,x,y,z\right)=\left(1,2,3,4\right)$ is a solution of first and fourth equations. By substitution, it also satisfies the second and third equations. Since the equations are symmetric, all permutations are solutions, that is

 $\left(w,x,y,z\right)=\left(1,2,3,4\right)=\left(1,2,4,3\right)=\left(1,3,2,4\right)=…\left(4,3,2,1\right)$

 However, these are all the solutions since the product of the degrees of the equations is $4!=24.$

**3.** Given:

$$\left\{\begin{matrix}a^{2}+b^{2}+ab=9\\b^{2}+c^{2}+bc=16\\c^{2}+a^{2}+ca=25\end{matrix}\right.$$

 **(a)** If $a,b,c>0$ , find $ab+bc+ca$.

 **(b)** **(Hard)** If $a,b,c$ are real numbers, find $ab+bc+ca$.

 (a) Put
$$\left\{\begin{matrix}p^{2}=a^{2}+b^{2}+ab=a^{2}+b^{2}-2ab\cos(120°)=9\\q^{2}=b^{2}+c^{2}+bc=b^{2}+c^{2}-2bc\cos(120°)=16\\r^{2}=c^{2}+a^{2}+ca=c^{2}+a^{2}-2ca\cos(120°)=25\end{matrix}\right.$$

 Since $p^{2}+q^{2}=9+16=3^{2}+4^{2}=25=5^{2}=r^{2}, p=3,q=4,r=5$

 By the converse of Pythagoras Theorem, we can form $∆PQR$ right-angled at $R$.

 $M$ is a point inside $∆PQR$ such that

 $∠PMQ=∠QMR=∠RMP=120°$

 $Area of ∆PQR$

 $=Area of ∆RMP+Area of ∆QMR+Area of ∆PMQ$

 $\frac{1}{2}×3×4=\frac{1}{2}ab\sin(120°+\frac{1}{2}bc\sin(120°+)\frac{1}{2}ca\sin(120°))$

$$=\frac{1}{2}ab\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}bc\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}ca\left(\frac{\sqrt{3}}{2}\right)$$

$$∴ ab+bc+ca=6×\frac{4}{\sqrt{3}}=\overline{\overline{8\sqrt{3}}}$$

**(b) (i)** If $a,b,c<0$ , let $x=-a,y=-b,z=-c$ then the equation becomes:

$$\left\{\begin{matrix}x^{2}+y^{2}-xy=9\\y^{2}+z^{2}-yz=16\\z^{2}+x^{2}-zx=25\end{matrix}\right.$$

 which is similar to the original set of equation,

$∴ xy+yz+zx=ab+bc+ca=\overline{\overline{8\sqrt{3}}}$

 **(ii)** If $a,c>0, b<0$ , let $x=a,y=-b,z=c$ then the equation becomes:

$$\left\{\begin{matrix}x^{2}+y^{2}-xy=9\\y^{2}+z^{2}-yz=16\\z^{2}+x^{2}+zx=25\end{matrix}\right.$$

 Put
$$\left\{\begin{matrix}p^{2}=x^{2}+y^{2}-xy=x^{2}+y^{2}-2xy\cos(60°)=9\\q^{2}=y^{2}+z^{2}-yz=y^{2}+z^{2}-2yz\cos(60°)=16\\r^{2}=z^{2}+x^{2}+zx=z^{2}+x^{2}-2zx\cos(120°)=25\end{matrix}\right.$$

 Since $p^{2}+q^{2}=9+16=3^{2}+4^{2}=25=5^{2}=r^{2}, p=3,q=4,r=5$

 By the converse of Pythagoras Theorem, we can form $∆PQR$ right-angled at $R$.

 $M$ is a point outside $∆PQR$ such that

 $∠PMQ=120°, ∠QMR=∠RMP=60°$

 $Area of ∆PQR$

 $=Area of ∆RMP+Area of ∆QMR-Area of ∆PMQ$

 $\frac{1}{2}×3×4=\frac{1}{2}xy\sin(60°+\frac{1}{2}yx\sin(60°-)\frac{1}{2}zx\sin(120°))$

$$=\frac{1}{2}xy\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}yz\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}zx\left(\frac{\sqrt{3}}{2}\right)$$

$$=-\frac{1}{2}ab\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}bc\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}ca\left(\frac{\sqrt{3}}{2}\right)$$

$$∴ ab+bc+ca=-6×\frac{4}{\sqrt{3}}=\overline{\overline{-8\sqrt{3}≈-13.856406460551}}$$

 **(iii)** If $a<0, b,c>0$ , let $x=a,y=-b,z=-c$

 we can use similar method as in **(ii)**. (Readers may try.)

 and can get $ab+bc+ca=\overline{\overline{-8\sqrt{3}≈-13.856406460551}}$

 **(iv)** We don’t have solutions for other cases such as $a,b>0, c<0$ .

**Tough readers may also try to find the values for a, b, c.**

**I include here the complete solution.**









$$where d=ab+bc+ca$$

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**4. (Hard)** Solve for real numbers $a,b,c$ :

$$\left\{\begin{array}{c}a^{2}-2ab+bc-c^{2}+ca=0\\b^{2}-2bc+ca-b^{2}+ab=0\\c^{2}-2ca+ab-a^{2}+bc=0\end{array}\right.$$

 The system is cyclic, without loss of generality, let $a\geq b\geq c$.

 If $b=c$ , then

 $\left\{\begin{array}{c}a^{2}-2ab+bc-c^{2}+ca=0\\b^{2}-2bc+ca-b^{2}+ab=0\\c^{2}-2ca+ab-a^{2}+bc=0\end{array}\right.\begin{matrix}\\⟹\\\end{matrix}\left\{\begin{array}{c}a^{2}-2ab+b^{2}-b^{2}+ab=0\\b^{2}-2b^{2}+ab-b^{2}+ab=0\\b^{2}-2ba+ab-a^{2}+b^{2}=0\end{array}\right.\begin{matrix}\\⟹\\\end{matrix}\left\{\begin{array}{c}a^{2}-ab=0\\2ab-2b^{2}=0\\-a^{2}-ab+2b^{2}=0\end{array}\right.$

 $\begin{matrix}\\⟹\\\end{matrix}\left\{\begin{array}{c}a(a-b)=0\\2b(a-b)=0\\\left(2b+a\right)(b-a)=0\end{array}\right.\begin{matrix}\\⟹\\\end{matrix} a=b$ , ($a,b$ may be 0)

 Therefore $a=b=c$

 Putting $a=b=c$ in the original system confirms the result.

 Since the system is cyclic, if $a=b$, we can still give $a=b=c$ as solution.

 Thus we can assume distinct values of $a,b,c$, that is, $a>b>c$ .

 Then we can replace $a=b+x, b=c+y, x,y>0 ∴a=c+x+y, $

$$\left\{\begin{array}{c}a^{2}-2ab+bc-c^{2}+ca=0\\b^{2}-2bc+ca-b^{2}+ab=0\\c^{2}-2ca+ab-a^{2}+bc=0\end{array}\right.$$

$$\left\{\begin{array}{c}\left(c+x+y\right)^{2}-2\left(c+x+y\right)\left(c+y\right)+\left(c+y\right)c-c^{2}+c\left(c+x+y\right)=0\\\left(c+y\right)^{2}-2\left(c+y\right)c+c\left(c+x+y\right)-\left(c+y\right)^{2}+\left(c+x+y\right)\left(c+y\right)=0\\c^{2}-2c\left(c+x+y\right)+\left(c+x+y\right)\left(c+y\right)-\left(c+x+y\right)^{2}+\left(c+y\right)c=0\end{array}\right.$$

$$\left\{\begin{array}{c}x^{2}+c x-y^{2}=0\\x y+2 c x+y^{2}+c y=0\\-x^{2}-x y-3 c x-c y=0\end{array}\right.$$

$$\left\{\begin{array}{c}c=-\frac{x^{2}-y^{2}}{x}, where x\ne 0….(1)\\c=-\frac{x y+y^{2}}{2 x+y}, where y\ne -2x….(2)\\c=-\frac{x^{2}+x y}{3 x+y}, where y\ne -3x….(3)\end{array}\right.$$

 So if $x\ne 0, y\ne -2x, y\ne -3x$ (Note that $x,y>0$, this holds.)

 $\left(1\right)=\left(2\right),$ $\frac{x^{2}-y^{2}}{x}=\frac{x y+y^{2}}{2 x+y}$ (When we put $\left(2\right)=\left(3\right)$, we get the same equation.)

$$\left(x^{2}-y^{2}\right)\left(2 x+y\right)=x\left(x y+y^{2}\right)$$

$$\left(x+y\right)\left(x-y\right)\left(2 x+y\right)=xy\left(x+y\right)$$

$$\left(x+y\right)\left(x-y\right)\left(2 x+y\right)-xy\left(x+y\right)=0$$

$$\left(x+y\right)\left[\left(x-y\right)\left(2 x+y\right)-xy\right]=0$$

$$\left(x+y\right)\left[2x^{2}-2xy-y^{2}\right]=0$$

 (a) If $x+y=0$ , then $x=-y$, but $x,y>0$ . The solution is rejected.

 (b) If $2x^{2}-2xy-y^{2}=0 or y^{2}+2xy-2x^{2}=0$

 Using quadratic equation formula, $y=-x\pm \sqrt{3x^{2}}$

 But since $x,y>0,$ we have $y=\left(\sqrt{3}-1\right)x ….(4)$

 $\left(4\right)\downright (1)$, $c=-\frac{x^{2}-\left(\sqrt{3}-1\right)^{2}x^{2}}{x}=\left(3-2 \sqrt{3}\right) x$

$$b=c+y=\left(3-2 \sqrt{3}\right) x+\left(\sqrt{3}-1\right)x=\left(2-\sqrt{3}\right)x$$

$$a=b+x=\left(2-\sqrt{3}\right)x+x=\left(3-\sqrt{3}\right)x$$

$\left(a,b,c\right)=\left(\left(3-\sqrt{3}\right)x,\left(2-\sqrt{3}\right)x,\left(3-2 \sqrt{3}\right) x\right)$, where $x$ is a free parameter.

**Complete solution:**

$\left(a,b,c\right)=\left(λ,λ.λ\right) or \left(\left(3-\sqrt{3}\right)μ,\left(2-\sqrt{3}\right)μ,\left(3-2 \sqrt{3}\right) μ\right)$ , where $λ, μ$ are parameters.

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